

NON-CROSSING LINKED PARTITIONS AND MULTIPLICATION OF FREE RANDOM VARIABLES

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ABSTRACT. The material gives a new combinatorial proof of the multiplicative property of the S -transform. In particular, several properties of the coefficients of its inverse are connected to non-crossing linked partitions and planar trees. AMS subject classification: 05A10 (Enumerative Combinatorics); 46L54 (Free Probability and Free Operator Algebras).

1. INTRODUCTION AND DEFINITIONS

The relation between non-crossing partitions and free probabilities has been studied extensively (see [5], [8]), but the closely related non-crossing linked partition have not received the same attention. Recently (see [2], [7]), the latest object was shown to give the recurrence for computing the coefficient of the inverse of the Voiculescu's S -transform in a similar manner the non-crossing partitions are used for the computation of the R -transform (see (1) and (2) below). The present material gives a new, combinatorial proof of the multiplicative property of the S -transform using a relation between planar rooted trees and the Kreweras complement.

A non-commutative probability space is a couple (\mathcal{A}, ϕ) , where \mathcal{A} is a unital $(*)$ -algebra and $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear mapping such that $\phi(1) = 1$ and $\phi(x^*x) \geq 0$ for all $x \in \mathcal{A}$ if \mathcal{A} is a $*$ -algebra. The $(*)$ -subalgebras $\{\mathcal{A}_i\}_{i \in I}$ of \mathcal{A} are *free* if

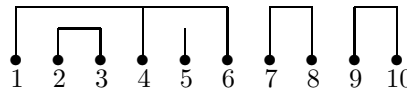
$$\phi(a_1 a_2 \cdots a_n) = 0$$

for any $a_k \in \mathcal{A}_{\varepsilon(k)}$ such that $\varepsilon(j) \neq \varepsilon(j+1)$ and $\phi(a_k) = 0$. The elements $\{X_i\}_{i \in I}$ from \mathcal{A} are free if the unital $*$ -algebras they generate are free.

A non-crossing partition γ of the ordered set $\{1, 2, \dots, n\}$ is a collection C_1, \dots, C_k of subsets of $\{1, 2, \dots, n\}$, called blocks, with the following properties:

- (a) $\bigsqcup_{l=1}^k C_l = \{1, \dots, n\}$ (disjoint union of sets)
- (b) C_1, \dots, C_k are non-crossing, in the sense that there are no two blocks C_l, C_s and $i < k < p < q$ such that $i, p \in C_l$ and $k, q \in C_s$.

Example 1: Below is represented graphically the non-crossing partition $\pi = (1, 4, 6), (2, 3), (5), (7, 8), (9, 10), (11, 12) \in NCL(10)$:



Non-crossing partitions appear in the definition of the free cumulants, the multilinear functions $\kappa_n : \mathcal{A}^n \longrightarrow \mathbb{C}$ given by the recurrence $(X_1, \dots, X_n \in \mathcal{A})$:

$$(1) \quad \phi(X_1 \cdots X_n) = \sum_{\gamma \in NC(n)} \prod_{\substack{C = \text{block in } \gamma \\ C = (i_1, \dots, i_l)}} \kappa_l(X_{i_1}, \dots, X_{i_l})$$

If $X_1 = \cdots = X_n = X$, then we write $\kappa_m(X)$ for $\kappa_m(X_1, \dots, X_n)$. A remarkable property of the free cumulants is the following:

Proposition 1.1. *If $\{\mathcal{A}_i\}_{i \in I}$ is a family of unital subalgebras of \mathcal{A} , then the following statements are equivalent:*

- (i) $\{\mathcal{A}_i\}_{i \in I}$ are free independent.
- (ii) For all $a_k \in \mathcal{A}_{i(k)}$ ($k = 1, \dots, n$ and $i(k) \in I$) we have that $\kappa_n(a_1, \dots, a_n) = 0$ whenever there exist $1 \leq l, k \leq n$ with $i(l) \neq i(n)$.

An immediate consequence is the additive property of the Voiculescu's R -transform (see [5], [3], [9]):

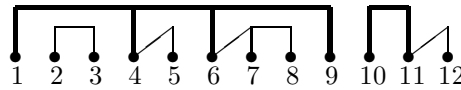
Proposition 1.2. *If, for any $a \in \mathcal{A}$, we define $R_a(z) = \sum_{n=1}^n \kappa_n(X) z^n$, then, for X, Y free, we have that $R_{X+Y} = R_X + R_Y$.*

By a non-crossing linked partition π of the ordered set $\{1, 2, \dots, n\}$ we will understand a collection B_1, \dots, B_k of subsets of $\{1, 2, \dots, n\}$, called blocks, with the following properties:

- (a) $\bigcup_{l=1}^k B_l = \{1, \dots, n\}$
- (b) B_1, \dots, B_k are non-crossing, in the sense that there are no two blocks B_l, B_s and $i < k < p < q$ such that $i, p \in B_l$ and $k, q \in B_s$.
- (c) for any $1 \leq l, s \leq k$, the intersection $B_l \cap B_s$ is either void or contains only one element. If $\{j\} = B_l \cap B_s$, then $|B_s|, |B_l| \geq 2$ and j is the minimal element of only one of the blocks B_l and B_s .

We will use the notation $s(\pi)$ for the set of all $1 \leq k \leq n$ such that there are no blocks of π whose minimal element is k . A block $B = i_1 < i_2 < \cdots < i_p$ of π will be called *exterior* if there is no other block D of π containing two elements l, s such that $l = i_1$ or $l < i_1 < i_p < s$. The set of all non-crossing linked partitions on $\{1, \dots, n\}$ will be denoted by $NCL(n)$.

Example 2: Below is represented graphically the non-crossing linked partition $\pi = (1, 4, 6, 9), (2, 3), (4, 5), (6, 7, 8), (10, 11), (11, 12) \in NCL(12)$. Its exterior blocks are $(1, 4, 6, 9)$ and $(10, 11)$.



Let $\mathcal{A}^\circ = \mathcal{A} \setminus \text{Ker } \phi$. Using non-crossing linked partitions, we define the t -coefficients $\{t_n\}_{n=0}^\infty$ as the mappings

$$t_n : \mathcal{A} \times (\mathcal{A}^\circ)^n \longrightarrow \mathbb{C}$$

given by the following recurrence:

$$(2) \quad \phi(X_1 \cdots X_n) = \sum_{\pi \in NCL(n)} \left(\prod_{\substack{B=\text{block in } \pi \\ B=(i_1, \dots, i_l)}} t_{l-1}(X_{i_1}, \dots, X_{i_l}) \cdot \prod_{k \in s(\pi)} t_0(X_k) \right).$$

To simplify the writing we will use the shorthand notations $t_\pi[X_1, \dots, X_n]$ for the summing term of the right-hand side of (2), and $t_\pi[X]$, respectively $t_n(X)$ for $t_\pi(X, \dots, X)$, respectively $t_n(X, \dots, X)$.

Remark 1.3. The mappings t_n are well-defined. Indeed, $t_{n-1}(X_1, \dots, X_n)$ appears only once and has a non-zero coefficient in the right hand side of (2), namely in

$$t_{\mathbb{1}_n}[X_1, \dots, X_n] = t_{n-1}(X_1, \dots, X_n) \prod_{l=2}^n t_0(X_l)$$

where $\mathbb{1}_n$ is the partition with a single block $(1, 2, \dots, n)$.

Also, while the free cumulants are multilinear, the t -coefficients have the property

$$t_n(c_0 X_0, c_1 X_1, \dots, c_n X_n) = c_0 t_n(X_0, X_1, \dots, X_n)$$

for all $c_0 \in \mathbb{C}, c_1, \dots, c_n \in \mathbb{C}^*$. The above relation is a immediate consequence of (2), since in each term of the right-hand side of (2), there is exactly one factor containing X_0 , on the first position in its block, and exactly two factors containing X_j ($j \geq 1$), among which only one has X_j in the first position.

An alternate form of (2) is described below. For $1 \leq k \leq n$ and $\pi \in NCL(n)$, we define

$$t_{[k, \pi]}(X_1, \dots, X_n) = \begin{cases} t_s(X_k, X_{i(1)}, \dots, X_{i(s)}) & \text{if } k \text{ is the minimal element} \\ & \text{of the block } (k, i(1), \dots, i(s)) \\ t_0(X_k) & \text{if } k \text{ is not the minimal element of any block.} \end{cases}$$

Then

$$\begin{aligned} t_\pi(X_1, \dots, X_n) &= \prod_{k=1}^n t_{[k, \pi]}(X_1, \dots, X_n) \\ \phi(X_1 \cdots X_n) &= \sum_{\pi \in NCL(n)} t_\pi(X_1, \dots, X_n). \end{aligned}$$

As shown in [2] and [7], the T -transform, defined via $T_X = \sum_{n=0}^{\infty} t_n(X) z^n$, has the property that $T_{XY} = T_X T_Y$ for all X, Y free elements of \mathcal{A}° . Notable is also the role of $NCL(n)$ in defining a conditionally free version of the T -transform (see [7]). In the following sections we will discuss the lattice structure of $NCL(n)$, prove a property similar to Proposition 1.1 for the t -coefficients and give a proof for the multiplicative property of the T -transform based on the connection between $NCL(n)$, the Kreweras complement on $NC(n)$ and planar rooted trees.

2. THE LATTICE $NCL(n)$

On $NCL(n)$ we define an order relation by saying that $\pi \succeq \sigma$ if for any block B of π there exist D_1, \dots, D_s blocks of σ such that $B = D_1 \cup \dots \cup D_s$. With respect to the order relation \succeq , the set $NCL(n)$ is a lattice. The maximal, respectively the

minimal element are $\mathbb{1}_n = (1, 2, \dots, n)$ and $0_n = (1), (2), \dots, (n)$. Note also that $NC(n)$ is a sublattice of $NCL(n)$.

We say that i and j are *connected* in $\pi \in NCL(n)$ if there exist B_1, \dots, B_s blocks of π such that $i \in B_1$, $j \in B_s$ and $B_k \cap B_{k+1} \neq \emptyset$, $1 \leq k \leq s-1$.

To $\pi \in NCL(n)$ we assign the partition $c(\pi) \in NC(n)$ defined as follows: i and j are in the same block of $c(\pi)$ if and only if they are connected in π . (I. e. the blocks of $c(\pi)$ are exactly the connected components of π .) We will use the notation

$$[c(\pi)] = \{\sigma \in NCL(n) : c(\sigma) = c(\pi)\}.$$

In the above Example 2, we have that 5 and 8 as well as 10 and 12 are connected. More precisely, $c(\pi) = (1, 4, 5, 6, 7, 8, 9), (2, 3), (10, 11, 12)$.

From the definition of the order relation \succeq , we have that, for every $\gamma \in NC(n)$, $[\gamma]$ is a sublattice of $NCL(n)$ and its maximal element is γ . Moreover, if γ has the blocks B_1, \dots, B_s , each B_l of cardinality k_l , then we have the following ordered set isomorphism:

$$(3) \quad [c(\pi)] \simeq [\mathbb{1}_{k_1}] \times \dots \times [\mathbb{1}_{k_s}]$$

The above factorization has two immediate consequences:

Proposition 2.1. *For any positive integer n and any $X_1, \dots, X_n \in \mathcal{A}$ we have that*

$$\kappa_n(X_1, \dots, X_n) = \sum_{\pi \in [\mathbb{1}_n]} t_\pi[X_1, \dots, X_n]$$

Proof. For $n = 1$ the assertion is clear, since, by definition,

$$\phi(X_1) = \kappa_1(X_1) = t_0(X_1).$$

If $n > 1$, note first that

$$\sum_{\pi \in NCL(n)} t_\pi[X_1, \dots, X_n] = \sum_{\gamma \in NC(n)} \sum_{\pi \in [\gamma]} t_\pi[X_1, \dots, X_n].$$

Also, if $\pi \in NCL(n)$ has the connected components B_1, \dots, B_k such that each $B_l = (i_{l,1}, \dots, i_{l,s(l)})$, then

$$t_\pi[X_1, \dots, X_n] = \prod_{l=1}^k t_{\pi|_{B_l}}[X_{i_{l,1}}, \dots, X_{i_{l,s(l)}}],$$

where $\pi|_{B_l}$ denotes the restriction of π to the set B_l .

Since the blocks of $c(\pi)$ are by definition the connected components of π , the relation (2) becomes:

$$\phi(X_1 \cdots X_n) = \sum_{\gamma \in NC(n)} \prod_{\substack{B=\text{block in } \gamma \\ B=(i_1, \dots, i_s)}} \sum_{\pi \in [\gamma]} t_{\pi|_B}[X_{i_1}, \dots, X_{i_s}],$$

and the factorization (3) gives:

$$\phi(X_1 \cdots X_n) = \sum_{\gamma \in NC(n)} \prod_{\substack{B=\text{block in } \gamma \\ B=(i_1, \dots, i_s)}} \sum_{\sigma \in [\mathbb{1}_s]} t_\sigma[X_{i_1}, \dots, X_{i_s}].$$

The conclusion follows now utilizing (1) and induction on n . □

Proposition 2.2. (Characterization of freeness in terms of t -coefficients)

If $\{\mathcal{A}_i\}_{i \in I}$ is a family of unital subalgebras of \mathcal{A} , then the following statements are equivalent:

- (i) $\{\mathcal{A}_i\}_{i \in I}$ are free independent.
- (ii) For all $a_k \in \mathcal{A}_{i(k)}$ ($k = 1, \dots, n$) and $i(k) \in I$ we have that

$$t_{n-1}(a_1, \dots, a_n) = 0$$

whenever there exist $1 \leq l, k \leq n$ with $i(l) \neq i(k)$.

Proof. It suffices to show the equivalence between 2.2(ii) and 1.1(ii).

Let first suppose that 2.2(ii) holds true. Choose a positive integer n and $\pi \in [\mathbb{1}_n]$. If not all X_k are coming from the same subalgebra \mathcal{A}_j , since $1, \dots, n$ are all connected in π , there is a block $B = (i_1, \dots, i_s)$ such that $\{X_{i_1}, \dots, X_{i_s}\}$ contains elements from different subalgebras, therefore $t_{s-1}(X_{i_1}, \dots, X_{i_s}) = 0$, hence $t_\pi[X_1, \dots, X_n] = 0$ and 2.1 implies that $\kappa_n(x_1, \dots, X_n) = 0$, i. e. 1.1(ii).

Suppose now that 1.1(ii) holds true. Proposition 2.1 implies that $\kappa_2 = t_1$, so 2.2(ii) is true for $n = 2$. An inductive argument on n will complete the proof:

Proposition 2.1 also implies that

$$\kappa_n(X_1, \dots, X_n) = t_{n-1}(X_1, \dots, X_n) + \sum_{\substack{\pi \in [\mathbb{1}_n] \\ \pi \neq \mathbb{1}_n}} t_\pi[X_1, \dots, X_n].$$

If not all X_k are coming from the same subalgebra \mathcal{A}_j , then the left-hand side of the above equation cancels from 1.1(ii) and so does the second term of the right-hand side, from the induction hypothesis, so q.e.d.. \square

3. PLANAR TREES AND THE MULTIPLICATIVE PROPERTY OF THE T -TRANSFORM

In this section we will give a combinatorial proof for the multiplicative property of the T -transform, that is

$$(4) \quad T_{XY} = T_X T_Y \text{ whenever } X \text{ and } Y \text{ are free elements from } \mathcal{A}^\circ.$$

The proof will consist mainly in describing certain bijections between $[\mathbb{1}_n]$ and the set $\mathfrak{T}(n)$ of *planar trees* with n vertices, respectively between $NC(n)$ and the set $\mathfrak{B}(n)$ of *bicolor planar trees* with n vertices.

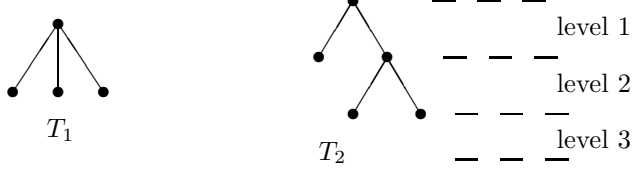
3.1. Non-crossing linked partitions and planar trees. By an *elementary planar tree* we will understand a graph with $m \geq 1$ vertices, v_1, v_2, \dots, v_m , and $m - 1$ edges, or branches, connecting v_1 (called *root*) to the vertices v_2, \dots, v_m (called *offsprings*). By convention, a single vertex (with no offsprings) will be also considered an elementary planar tree.

A *planar tree* will be seen as consisting in a finite number of *levels*, such that:

- first level consists in a single elementary planar tree, whose root will be also the root of the planar tree;
- the k -th level will consist in a set of elementary planar trees such that their roots are among the offsprings of the $k - 1$ -th level.

The set of elementary trees composing the planar tree A will be denoted by $E(A)$.

Below are represented graphically the elementary planar tree T_1 and the 2-level planar tree T_2 :

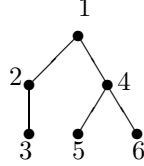


We will need to consider on the vertices of a planar tree the order relation (similar to the “*left depth first*” order from [1]) given by:

- (i) roots are less than their offsprings;
- (ii) offsprings of the same root are ordered from left to right;
- (iii) if v is less than w , then all the offsprings of v are smaller than any offspring of w .

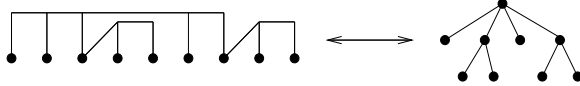
Intuitively, one may understand it as the order in which the vertices are passed by walking along the branches from the root to the right-most vertex, not counting vertices passed more than one time (see the example below).

Example 3:



We construct now the bijection $\Theta : [\mathbb{1}_n] \longrightarrow \mathfrak{T}(n)$ by putting $\Theta(\pi)$ be the planar tree composed by the elementary trees of vertices numbered (i_1, \dots, i_s) (with respect to the above order relation), for each (i_1, \dots, i_s) block of π .

More precisely, if $(1, 2, i_1, \dots, i_s)$ is the block of π containing 1, then the first level of $\Theta(\pi)$ is the elementary planar tree of root numbered 1 and offsprings numbered $(2, i_1, \dots, i_s)$. The second level of $\Theta(\pi)$ will be determined by the blocks (if any) having $2, i_1, \dots, i_s$ as first elements etc (see the example below)



It is easy to see that Θ is well-defined and injective. To show that Θ is bijective, it suffices to observe that Θ^{-1} is given by assigning to each tree the partition having the blocks given by the numbers of the vertices from the constituent elementary planar trees. Part (a) of the definition of $NCL(n)$ (see Section 1) is automatically verified, since the trees have exactly n vertices. Part (b) follows from the conditions (ii) and (iii) in the definition of the order relation on the vertices, and part (c) from the conditions (i) and (ii).

If A_0 is an elementary planar tree with n vertices and $X \in \mathcal{A}^\circ$, we define $\mathcal{E}_X(A_0) = t_{n-1}(X)$. The evaluation \mathcal{E}_X extends to the set of planar trees by

$$\mathcal{E}_X(A) = \prod_{A_0 \in E(A)} \mathcal{E}_X(A_0).$$

Consequently,

$$(5) \quad \kappa_n(X) = \sum_{A \in \mathfrak{A}(n)} \mathcal{E}_X(A).$$

3.2. The Kreweras complement and bicolor planar trees. For $\gamma \in NC(n)$, its *Kreweras complement* $Kr(\gamma)$ is defined as follows. We consider the additional numbers $\bar{1}, \dots, \bar{n}$ forming the ordered set

$$1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}.$$

$Kr(\gamma)$ is defined to be the biggest element $\gamma' \in NC(\bar{1}, \dots, \bar{n}) \cong NC(n)$ such that

$$\gamma \cup \gamma' \in NC(1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}).$$

The total number of blocks in γ and $Kr(\gamma)$ is $n + 1$ (see [5], [4]). The Kreweras complement appears in the following corollary of Proposition 1.1:

Proposition 3.1. *If X, Y are free elements of \mathcal{A} , then*

$$\kappa_n(XY) = \sum_{\gamma \in NC(n)} \kappa_\gamma[X] \kappa_{Kr(\gamma)}[Y].$$

Let $NC_s(2n)$ be the set of all $\gamma \in NC(2n)$ such that elements from the same block of γ have the same parity and $\gamma_+ = Kr(\gamma_-)$, where

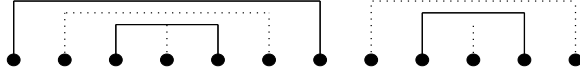
$$\begin{aligned} \gamma_+ &= \gamma|_{\{2, 4, \dots, 2n\}} \\ \gamma_- &= \gamma|_{\{1, 3, \dots, 2n-1\}}. \end{aligned}$$

Denote also $NCL_S(2n) = \{\pi \in NCL(2n) : c(\pi) \in NC_s(2n)\}$. With the above notations, the relation from Proposition 3.1 becomes:

$$(6) \quad \kappa_n(XY) = \sum_{\gamma \in NC_s(2n)} \kappa_{\gamma_-}[X] \kappa_{\gamma_+}[Y]$$

For $\pi \in NC_s(2n)$, we will say that the blocks with odd elements are of color 1 and the ones with even elements are of color 0. Note that $\pi \in NC_s(2n)$ if and only if π has exactly 2 exterior blocks, one of color 1 and one of color 0 and if i_1 and i_2 are two consecutive elements from the same block, then $\pi|_{(i_1+1, \dots, i_2-1)}$ has exactly one exterior block, of different color than the one containing i_1 and i_2 .

We will represent blocks of color 1 by solid lines and blocks of color 0 by dashed lines:



In the remaining part of this subsection we will define the set $\mathfrak{B}(n)$ of bicolor planar trees and construct a bijection $\Lambda : NC_S(2n) \longrightarrow \mathfrak{B}(n)$.

A *bicolor elementary planar tree* is an elementary tree together with a mapping from its offsprings to $\{0, 1\}$ such that the offsprings whose image is 1 are smaller (in the sense of Section 3.1) than the offsprings of image 0. Branches toward offsprings of color 0, respectively 1, will be also said to be of color 0, respectively 1. We will represent by solid lines the branches of color 1 and by dashed lines the branches of color 0. The set of all bicolor planar trees with n vertices will be denoted by $\mathfrak{EB}(n)$. Below is the graphical representation of $\mathfrak{EB}(4)$:

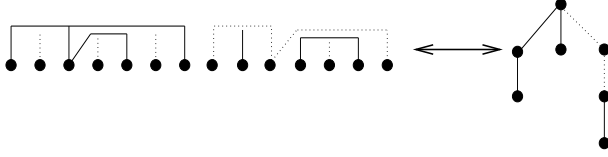


A *bicolor planar tree* is a planar tree whose constituent elementary trees are all bicolor; the set of all bicolor planar trees will be denoted by $\mathfrak{B}(n)$.

Given $\pi \in NCL_S(2n)$, we construct $\Lambda(\pi) \in \mathfrak{B}(n)$ as follows:

- If (i_1, \dots, i_s) and (j_1, \dots, j_p) are the two exterior blocks of π , then the first level of $\Lambda(\pi)$ is an elementary tree with $s - 1 + p - 1$ offsprings, the first $s - 1$ of color 1, corresponding to (i_2, \dots, i_s) , in this order, and the last $p - 1$ of color 0, corresponding to (j_2, \dots, j_p) , in this order.
- Suppose that i_1 and i_2 are consecutive elements in a block of π already represented in an elementary tree of $\Lambda(\pi)$, that π has the exterior block (j_1, \dots, j_p) and that i_2 is the minimal element of the block (i_2, d_1, \dots, d_r) . The blocks $B = (j_1, \dots, j_p)$ and $D = (i_2, d_1, \dots, d_r)$ will have different colors. They will be then represented by an elementary tree of vertex corresponding to i_2 (the block of i_1 and i_2 has been already represented from the hypothesis), and with $p - 1 + k$ offsprings, keeping the colors of the blocks B and D , the ones of color 1 placed before the ones of color 0.

Note that the mapping Λ is bijective, the inverse is constructing reversing the steps above.



Fix $X, Y \in \mathcal{A}^\circ$, free. If $B_0 \in \mathfrak{E}\mathfrak{B}(n)$ has k offsprings of color 1 and $n - k - 1$ offsprings of color 0, then we define

$$\omega_{X,Y}(B_0) = t_k(X)t_{n-k-1}(Y).$$

The functional $\omega_{X,Y}$ extends to $B \in \mathfrak{B}(n)$ via

$$\omega_{X,Y}(B) = \prod_{B_0 \in E(B)} \omega_{X,Y}(B_0).$$

For $\pi \in NC_S(2n)$ the definition of the mapping Λ gives

$$(7) \quad \kappa_{\pi_-}[X]\kappa_{\pi_+}[Y] = \omega_{X,Y}(\Lambda(\pi))$$

Theorem 3.2. *If X, Y are free elements from \mathcal{A}° , then $T_{XY} = T_X T_Y$.*

Proof. We need to show that, for all $m \geq 0$

$$(8) \quad t_m(XY) = \sum_{k=0}^m t_k(X)t_{m-k}(Y)$$

For $m = 0$, the assertion is trivial. Suppose (8) true for $m \leq n - 1$. Let A_m be the elementary planar tree with m vertices. In terms of planar trees, the induction hypothesis is written as

$$(9) \quad \mathcal{E}_{XY}(A_m) = \sum_{B \in \mathfrak{E}\mathfrak{B}(n)} \omega_{X,Y}(B).$$

For example,

$$\mathcal{E}_{XY}(\text{tree with 3 vertices}) = \omega_{XY}(\text{tree 1}) + \omega_{XY}(\text{tree 2}) + \omega_{XY}(\text{tree 3}) + \omega_{XY}(\text{tree 4})$$

The relations (5) and (7) give:

$$\begin{aligned}
 \sum_{A \in \mathfrak{T}(n)} \mathcal{E}_{XY}(A) &= \kappa_n(X) \\
 &= \sum_{\pi \in NC_S(2n)} \kappa_{\pi_-}[X] \kappa_{\pi_+}[Y] \\
 &= \sum_{\pi \in NC_S(2n)} \omega_{X,Y}(\Lambda(\pi)) \\
 (10) \quad &= \sum_{B \in \mathfrak{B}(n)} \omega_{X,Y}(B).
 \end{aligned}$$

All non-elementary trees from $\mathfrak{T}(n)$ consists on elementary trees with less than n vertices. The relation (9) implies that the image under \mathcal{E}_{XY} of any such tree is the sum of the images under ω_{XY} of its colored versions. Hence

$$(11) \quad \sum_{\substack{A \in \mathfrak{T}(n) \\ A \neq A_n}} \mathcal{E}_{XY}(A) = \sum_{\substack{B \in \mathfrak{B}(n) \\ B \in \mathfrak{E}\mathfrak{B}(n)}} \omega_{X,Y}(B)$$

Finally (11) and (10) give

$$\mathcal{E}_{XY}(A_n) = \sum_{B \in \mathfrak{B}(n)} \omega_{X,Y}(B)$$

that is (8). □

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